On some multiplicative product of kth derivative of Dirac's delta in $x_0 + |x|$ and $x_0 - |x|$

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In this paper we give a sense to the products

$$\frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}}$$

and $\delta^{(k-1)}(x_0 - |x|) \cdot \delta^{(k-1)}(x_0 + |x|)$. The first of them is a generalization of the product

$$\frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}}$$

given in [1, p. 158].

1. Introduction

Let $P(x_1, x_2, \ldots, x_n)$ be any sufficiently smooth function such that on $P = P(x_1, x_2, \ldots, x_n) = 0$ we have $\operatorname{grad} P \neq 0$ (which means that there are no singular points on $P = 0$). Then the generalized function $\delta^{(k)}P$ is defined in [4, p. 211] by

$$\left\langle \delta^{(k)(P)}, \varphi \right\rangle = (-1)^k \int \psi^{(k)}_{u_1}(0, u_2, \ldots, u_n) \, \mathrm{d}u_2 \ldots \mathrm{d}u_n, \tag{1}$$

where

$$u_1 = P \tag{2}$$

and choose the remaining u_i coordinates (with $i = 2, \ldots, n$) arbitrarily except that the Jacobian of the x_i with respect to the u_i which we shall denote by $D\binom{x}{u}$ fails to vanish (which is always possible so long as $\operatorname{grad} P \neq 0$ and $P = 0$).

In (1)

$$\psi(u_1, u_2, \ldots, u_n) = \varphi(u_1, u_2, \ldots, u_n) D\begin{pmatrix} x \\ u \end{pmatrix}, \tag{3}$$

$$\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, x_2, \ldots, x_n), \tag{4}$$

and the integral of (1) is taken over the $P = 0$ surface.

From [1, formula (1.5), p. 149] and [1, formula (1.16), p. 151]

$$P(x_0, x_1, \ldots, x_{n-1}) = x_0 - |x| \tag{5}$$

and for

$$P(x_0, x_1, \ldots, x_{n-1}) = x_0 + |x| \tag{6}$$

we have

$$\left\langle \delta^{(k-1)}\left(x_0 - |x|\right), \varphi \right\rangle = \int_0^\infty \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \left\{ \psi(x_0, s) s^{n-2} \right\} \right]_{s=x_0} \mathrm{d}x_0 \tag{7}$$

and

$$\left\langle \delta^{(k-1)}\left(x_0 + |x|\right), \varphi \right\rangle = (-1)^{k-1} \int_0^{-\infty} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \left\{ \psi(x_0, s) s^{n-2} \right\} \right]_{s=-x_0} \mathrm{d}x_0, \tag{8}$$

where

$$\psi(x_0, s) = \int_\Omega \varphi \, \mathrm{d}\Omega_{n-1} \tag{9}$$

and $\mathrm{d}\Omega_{n-1}$ is the element of surface area on the united sphere in $\mathbb{R}^{n-1}$.

On the other hand, and from [1, formula (2.5)], we have

$$\left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi \right\rangle = \int_0^\infty \int_0^{x_0} (x_0 - s)^\lambda \psi(x_0, s) s^{n-2} \, \mathrm{d}s \, \mathrm{d}x_0. \tag{10}$$

Now, make the change of variables

$$s = x_0 l \tag{11}$$

in the integral (10), writing

$$\psi(x_0, s) = \psi_1(x_0, x_0 l) \tag{12}$$

to obtain

$$\left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi \right\rangle = \int_0^\infty \int_0^1 x_0^{\lambda+n-1} (1 - l)^\lambda l^{n-2} \psi_1(x_0, x_0 l) \, \mathrm{d}l \, \mathrm{d}x_0. \tag{13}$$

This equation shows that $\langle (x_0 - |x|)_+^\lambda, \varphi \rangle$ has two poles. The first of these consists of the poles of

$$G(\lambda, x_0) = \int_0^1 (1 - l)^\lambda l^{n-2} \psi_1(x_0, x_0 l) \, \mathrm{d}l. \tag{14}$$

Using that [4, p. 49]

$$\operatorname*{Res}_{\lambda=-k} \left\langle x_+^\lambda, \varphi \right\rangle = \frac{\varphi^{(k-1)}(0)}{(k-1)!}, \quad k = 1, 2, \ldots, \tag{15}$$

from (14) we have

$$\operatorname*{Res}_{\lambda=-k} G(\lambda, x_0) = \frac{(-1)^{k-1}}{(k-1)!}\left[\frac{\partial^{k-1}}{\partial l^{k-1}}\left\{l^{n-2}\psi_1(x_0, x_0 l)\right\}\right]_{l=1}, \quad k = 1, 2, \dots . \tag{16}$$

On the other hand,

$$\left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi\right\rangle = \int_0^\infty x_0^{\lambda+n-1} G(\lambda, x_0)\,\mathrm{d}x_0 \tag{17}$$

may also have poles. This occurs at $\lambda = -n, -n-1, -n-2, \dots$.

At these points

$$\operatorname*{Res}_{\lambda=-n-j} \left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi\right\rangle = \frac{1}{j!}\left[G^{(j)}(-n-j, x_0)\right]_{x_0=0}, \quad j = 0, 1, 2, \dots . \tag{18}$$

Consequently, $\langle (x_0 - |x|)_+^\lambda, \varphi\rangle$ has two sets of singularities, namely,

$$\lambda = -1, -2, \dots$$

and

$$\lambda = -n, -n-1, -n-2, \dots ,$$

where n is the dimension of the space.

Let us now study the case when

$$\lambda = -k, \quad k = 1, 2, \dots \tag{19}$$

and

$$\lambda \neq -n, -n-1, -n-2, \dots . \tag{20}$$

Let us write (14) in the neighborhood of $\lambda = -k$ in the form

$$G(\lambda, x_0) = \frac{G_0(x_0)}{\lambda + k} + G_1(\lambda, x_0), \tag{21}$$

where

$$G_0(x_0) = \operatorname*{Res}_{\lambda=-k} G(\lambda, x_0) \tag{22}$$

and $G_1(\lambda, x_0)$ is regular at $\lambda = -k$.

Inserting this into (17), we obtain

$$\left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi\right\rangle = \frac{1}{\lambda + k}\int_0^\infty x_0^{\lambda+n-1} G_0(x_0)\,\mathrm{d}x_0 + \int_0^\infty x_0^{\lambda+n-1} G_1(\lambda, x_0)\,\mathrm{d}x_0. \tag{23}$$

Under the assumptions we have made concerning λ, the integrals in (23) are regular functions of λ at $\lambda = -k$. Therefore, $\langle (x_0 - |x|)_+^\lambda, \varphi\rangle$ has a simple pole at such a point, and

$$\operatorname*{Res}_{\lambda=-k} \left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi\right\rangle = \int_0^\infty x_0^{n-k-1} G_0(x_0)\,\mathrm{d}x_0, \quad k = 1, 2, \dots ,$$

where for $k \geqslant n$ the integral is understood in the sense of its regularization (see [4, chapter I, section 3]).

Inserting equation (16) for $G_0(x_0)$, we arrive at

$$\operatorname*{Res}_{\lambda=-k} \langle (x_0 - |x|)_+^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty x_0^{n-k-1} \left[\frac{\partial^{k-1}}{\partial l^{k-1}} \{ l^{n-2} \psi_1(x_0, x_0 l) \} \right]_{l=1} \mathrm{d}x_0, \quad k = 1, 2, \ldots . \tag{24}$$

Note that if we write $x_0 l = s$, we obtain

$$\left[\frac{\partial^{k-1}}{\partial l^{k-1}} \{ l^{n-2} \psi_1(x_0, x_0 l) \} \right]_{l=1} = x_0^{k-1-n+2} \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \{ s^{n-2} \psi_1(x_0, s) \} \right]_{s=x_0}, \tag{25}$$

so that we may rewrite (24) in the form

$$\operatorname*{Res}_{\lambda=-k} \langle (x_0 - |x|)_+^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \{ s^{n-2} \psi_1(x_0, s) \} \right]_{s=x_0} \mathrm{d}x_0, \quad k = 1, 2, \ldots . \tag{26}$$

On the other hand, for the generalized function $(x_0 + |x|)_-^\lambda$ defined by

$$\langle (x_0 + |x|)_-^\lambda, \varphi \rangle = \int_{-(x_0+|x|)>0} \left(-(x_0 + |x|) \right)^\lambda \varphi(x) \, \mathrm{d}x \tag{27}$$

as in the case for $(x_0 - |x|)_+^\lambda$ we arrive at the following result analogous to (14) and (17):

$$\langle (x_0 + |x|)_-^\lambda, \varphi \rangle = \int_0^{-\infty} (-x_0)^{\lambda+n-1} G(\lambda, -x_0) \, \mathrm{d}x_0, \tag{28}$$

where

$$G(\lambda, -x_0) = \int_0^1 (1-l)^\lambda l^{n-2} \psi_1(x_0, -x_0 l) \, \mathrm{d}l. \tag{29}$$

From (28), (29) and considering (15)–(17) we have

$$\operatorname*{Res}_{\lambda=-j} G(\lambda, -x_0) = \frac{(-1)^{j-1}}{(j-1)!} \left[\frac{\partial^{j-1}}{\partial l^{j-1}} \{ l^{n-2} \psi_1(x_0, -x_0 l) \} \right]_{l=1}, \quad j = 1, 2, \ldots, \tag{30}$$

and

$$\operatorname*{Res}_{\lambda=-n-j} \langle (x_0 + |x|)_-^\lambda, \varphi \rangle = (-1) \frac{1}{j!} \left[G^{(j)}(-n-j, x_0) \right]_{x_0=0}, \quad j = 0, 1, 2, \ldots . \tag{31}$$

Also $\langle (x_0 + |x|)_-^\lambda, \varphi \rangle$ has two sets of singularities

$$\lambda = -1, -2, \ldots$$

and

$$\lambda = -n, -n-1, \dots , \tag{32}$$

and considering equations (21)–(25) we arrive at the following result analogous to (26):

$$\operatorname*{Res}_{\lambda=-j} \left\langle \left(x_0 + |x|\right)_-^\lambda, \varphi \right\rangle = \frac{(-1)^{j-1}}{(j-1)!} \int_0^{-\infty} \left[\frac{\partial^{j-1}}{\partial s^{j-1}} \left\{ s^{n-2} \psi_1(x_0, s) \right\} \right]_{s=-x_0} \mathrm{d}x_0, \quad j = 1, 2, \dots . \tag{33}$$

Summarizing, from (7) and (26) we arrive at the following formula:

$$\delta^{(k-1)}\left(x_0 - |x|\right) = \frac{(k-1)!}{(-1)^{k-1}} \operatorname*{Res}_{\lambda=-k} \left(x_0 - |x|\right)_+^\lambda \quad \text{if } k < n, \quad k = 1, 2, \dots , \tag{34}$$

and similarly from (8) and (33) we obtain

$$\delta^{(k-1)}\left(x_0 - |x|\right) = (k-1)! \operatorname*{Res}_{\lambda=-k} \left(x_0 + |x|\right)_-^\lambda \quad \text{if } k < n, \quad k = 1, 2, \dots . \tag{35}$$

On the other hand, from (34) and considering (10) we have

$$\begin{aligned} &\left\langle \delta^{(k-1)}\left(x_0 - |x|\right), \varphi \right\rangle \\ &\quad = \frac{(k-1)!}{(-1)^{k-1}} \operatorname*{Res}_{\lambda=-k} \left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi \right\rangle \\ &\quad = \frac{(k-1)!}{(-1)^{k-1}} \lim_{\lambda \to -k} (\lambda + k) \left\langle \left(x_0 - |x|\right)_+^\lambda, \varphi \right\rangle \\ &\quad = \frac{(k-1)!}{(-1)^{k-1}} \lim_{\lambda \to -k} (\lambda + k) \int_{x_0 - |x| > 0} \left(x_0 - |x|\right)^\lambda \varphi(x) \, \mathrm{d}x_0 \dots \mathrm{d}x_{n-1} \\ &\quad = \frac{(k-1)!}{(-1)^{k-1}} \lim_{\lambda \to -k} (\lambda + k) \int_0^\infty \int_0^{x_0} (x_0 - s)^\lambda s^{n-2} \psi(x_0, s) \, \mathrm{d}s \, \mathrm{d}x_0, \\ &\quad k = 1, 2, \dots . \end{aligned} \tag{36}$$

From (36) we have

$$\begin{aligned} &\left\langle \frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}}, \varphi \right\rangle \\ &\quad = \frac{(k-1)!}{(-1)^{k-1}} \lim_{\lambda \to -k} (\lambda + k) \int_0^\infty \int_0^{x_0} (x_0 - s)^\lambda s^{(n-2)/2} \psi(x_0, s) \, \mathrm{d}s \, \mathrm{d}x_0. \end{aligned} \tag{37}$$

On the other hand, from [1], formulas (2.6) and (2.21), we have

$$\left\langle \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right\rangle = \int_0^\infty \int_0^{x_0} (x_0 - s)^\lambda \psi(x_0, s) s^{(n-2)/2} \, \mathrm{d}s \, \mathrm{d}x_0 \tag{38}$$

and

$$\left\langle \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right\rangle = \frac{1}{\lambda + k} \int_0^\infty x_0^{\lambda + n/2} G_0(x_0) \, \mathrm{d}x_0 + \int_0^\infty x_0^{\lambda + n/2} G_1(x_0) \, \mathrm{d}x_0 \, . \tag{39}$$

From (37) and considering (38) and (39) we have

$$\left\langle \frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}}, \varphi \right\rangle = \frac{(k-1)!}{(-1)^{k-1}} \lim_{\lambda \to -k} (\lambda + k) \left\langle \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right\rangle$$
$$= \frac{(k-1)!}{(-1)^{k-1}} \operatorname*{Res}_{\lambda=-k} \left\langle \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right\rangle. \tag{40}$$

Similarly from (35) and considering (27)–(29) we have

$$\begin{aligned}
&\langle \delta^{(k-1)}(x_0 + |x|), \varphi \rangle \\
&= (k-1)! \operatorname*{Res}_{\lambda=-k} \langle (x_0 + |x|)_-^\lambda, \varphi \rangle \\
&= (k-1)! \lim_{\lambda \to -k} (\lambda + k) \langle (x_0 + |x|)_-^\lambda, \varphi \rangle \\
&= (k-1)! \lim_{\lambda \to -k} \int_{-(x_0+|x|)>0} \big(-(x_0 - |x|)\big)_+^\lambda \varphi(x_0, x_1, \ldots, x_{n-1})\, dx_0 \ldots dx_{n-1}, \\
&k = 1, 2, \ldots .
\end{aligned}$$

As in the case for $(x_0 - |x|)_+^\lambda$ we arrive at the formula

$$\left\langle \frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}}, \varphi \right\rangle = (k-1)! \operatorname*{Res}_{\lambda=-k} \left\langle \frac{(x_0 + |x|)_-^\lambda}{|x|^{(n-2)/2}}, \varphi \right\rangle. \tag{41}$$

Summarizing, from (40) and (41) we obtain the following formulae:

$$\frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}} = \frac{(k-1)!}{(-1)^{k-1}} \operatorname*{Res}_{\lambda=-k} \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}} \tag{42}$$

and

$$\frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}} = (k-1)! \operatorname*{Res}_{\lambda=-k} \frac{(x_0 + |x|)_-^\lambda}{|x|^{(n-2)/2}}. \tag{43}$$

From [1], we know that the product

$$\frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}}$$

exists and

$$\frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}} = \frac{1}{2} \pi^{(n-1)/2} \frac{1}{\Gamma((n-1)/2)} \delta(x_0, x_1, \ldots, x_{n-1}), \tag{44}$$

where

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2}.$$

In this paper we give a sense to the distributional multiplicative products

$$\frac{\delta^{(k-1)}(x_0-|x|)}{|x|^{(n-2)/2}}\cdot\frac{\delta^{(k-1)}(x_0+|x|)}{|x|^{(n-2)/2}}$$

and

$$\delta^{(k-1)}\big(x_0-|x|\big)\cdot\delta^{(k-1)}\big(x_0+|x|\big).$$

Our formula (59) is a generalization of equation (4.4) given in [1].

2. The multiplicative products $[\delta^{(k-1)}(x_0-|x|)/|x|^{(n-2)/2}]\cdot[\delta^{(k-1)}(x_0+|x|)/|x|^{(n-2)/2}]$ and $\delta^{(k-1)}(x_0-|x|)\cdot\delta^{(k-1)}(x_0+|x|)$

Consider

$$x_0^2-|x|^2=x_0^2-x_1^2-x_2^2-\cdots-x_{n-1}^2 \tag{45}$$

and consider the functional $(x_0^2-|x|^2)_-^\lambda$ defined by

$$\big\langle\big(x_0^2-|x|^2\big)_-^\lambda,\varphi\big\rangle=\int_{-(x_0^2-|x|^2)>0}\big(-\big(x_0^2-|x|^2\big)\big)^\lambda\varphi(x)\,\mathrm{d}x, \tag{46}$$

where $x=(x_0,x_1,\dots,x_{n-1})$ and

$$\mathrm{d}x=\mathrm{d}x_0\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1}. \tag{47}$$

From [4, p. 253] we know that integral (46) converges for $\mathrm{Re}(\lambda)\geqslant 0$ and is an analytic function of λ.

Analytic continuation to $\mathrm{Re}(\lambda)<0$ can be used to extend the definition of $\langle(x_0^2-|x|^2)_-^\lambda,\varphi\rangle$.

From [4, chapter III, section 2.2], $\langle(x_0^2-|x|^2)_-^\lambda,\varphi\rangle$ has two sets of singularities, namely,

$$\lambda=-1,-2,\dots,-k,\dots\quad\text{and}\quad\lambda=-n/2,-n/2-1,\dots,-n/2-k,\dots. \tag{48}$$

When

$$\lambda=-k\quad\text{and}\quad\lambda\neq-n/2,-n/2-1,\dots,-n/2-h,\dots,\quad h=0,1,2,\dots,$$

this is always the case when the dimension n is odd, but is also true if n is even and $k<n/2$.

Now, considering [4, p. 255], we have

$$\big\langle\big(x_0^2-|x|^2\big)_-^\lambda,\varphi\big\rangle=\frac{1}{\lambda+k}\int_0^\infty\nu^{\lambda+n/2-1}H_0(\nu)\,\mathrm{d}\nu+\int_0^\infty\nu^{\lambda+n/2-1}H_1(\lambda,\nu)\,\mathrm{d}\nu, \tag{49}$$

where

$$H_0(\nu) = \operatorname*{Res}_{\lambda=-k} H(\lambda,\nu), \tag{50}$$

$$H(\lambda,\nu) = \frac{1}{4}\int_0^1 (1-t)^\lambda t^{(n-3)/2}\psi_1(\nu,t\nu)\,\mathrm{d}t \tag{51}$$

and $H_1(\lambda,\nu)$ is regular at $\lambda=-k$.

In (51) $\psi_1(\nu,t\nu)=\psi_1(u,\nu)=\psi(x_0,s)$, where $\psi(x_0,s)$ is defined by (9).

On the other hand, taking into account the Laurent expansion of $(x_0^2-|x|^2)_-^\lambda$ about $\lambda=-n/2-k$, from [4, p. 269] we have

$$\left(x_0^2-|x|^2\right)_-^\lambda = \frac{c_{-2}^{(k)}}{(\lambda+n/2+k)^2} + \frac{c_{-1}^{(k)}}{(\lambda+n/2+k)} + \cdots \tag{52}$$

and

$$\lim_{\lambda\to -n/2-k}\left\langle(\lambda+n/2+k)^2\left(x_0^2-|x|^2\right)_-^\lambda,\varphi\right\rangle$$
$$= \left\langle c_{-2}^{(k)},\varphi\right\rangle = \frac{(-1)^{n/2}\pi^{n/2-1}}{2^{2k}k!\Gamma(n/2+k)}\Box^k\left\{\delta(x)\right\} \tag{53}$$

if n is even, where

$$\Box^k = \left\{\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_{n-1}^2}\right\}^k. \tag{54}$$

Now when considering the functional $(x_0^2-|x|^2)_-^\lambda/|x|^{n-2}$, from [1, pp. 157 and 158] we observe that $\langle(x_0^2-|x|^2)_-^\lambda/|x|^{n-2},\varphi\rangle$ has singularities at $\lambda=-j,\ j=1,2,\ldots,$ and

$$\lim_{\lambda\to -1}(\lambda+1)^2\frac{(x_0^2-|x|^2)_-^\lambda}{|x|^{n-2}} = \frac{1}{2}\pi^{(n-1)/2}\frac{1}{\Gamma((n-1)/2)}\delta(x_0,x_1,\ldots,x_{n-1}). \tag{55}$$

Also from [1, formula (4.3), p. 12] and [1, p. 158] the following properties are valid:

$$\left(x_0-|x|\right)_+^\lambda\cdot\left(x_0+|x|\right)_-^\lambda = \left(x_0^2-|x|^2\right)_-^\lambda, \tag{56}$$

where

$$\left(x_0-|x|\right)_+^\lambda = \begin{cases}\left(x_0-|x|\right)^\lambda & \text{if } x_0-|x|\geqslant 0,\\ 0 & \text{if } x_0-|x|<0,\end{cases} \tag{57}$$

and

$$\left(x_0+|x|\right)_-^\lambda = \begin{cases}\left(-\left(x_0+|x|\right)\right)^\lambda & \text{if } x_0+|x|\leqslant 0,\\ 0 & \text{if } x_0+|x|>0.\end{cases} \tag{58}$$

Theorem 1. Let k be a positive integer and n dimension of the space. Then under the conditions

(a) n odd,

(b) n even if $k < n/2$

the following formula is valid:

$$\frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}} = a_{k,n}\Box^{k-1}\big\{\delta(x_0, x_1, \ldots, x_{n-1})\big\}, \tag{59}$$

where

$$a_{k,n} = \frac{1}{2}\,\frac{(k-1)!(n/2-1)\Gamma(n/2-k)\pi^{(n-1)/2}}{2^{2(k-1)}\Gamma(n/2)\Gamma((n-1)/2)} \tag{60}$$

and $\Box^{k-1}$ is defined by (54).

Proof. From (42) and (43) and considering (54) we have

$$\begin{aligned}&\frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}}\\ &\quad = \frac{(k-1)1(k-1)1}{(-1)^{k-1}} \lim_{\lambda\to -k} (\lambda+k)^2 \frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}.\end{aligned} \tag{61}$$

Now, considering the formulae (25) and (28) from [4, pp. 258 and 259], we have

$$\begin{aligned}\big\langle \big(x_0^2 - |x|^2\big)_-^\lambda, \varphi\big\rangle &= \frac{1}{2^{2m}(\lambda+1)\cdots(\lambda+m)(\lambda+n/2)\cdots(\lambda+n/2+m-1)}\\ &\quad \times \big\langle \big(x_0^2 - |x|^2\big)_-^{\lambda+m}, \Box^m\varphi\big\rangle,\end{aligned} \tag{62}$$

where $\Box^k$ is defined by (54).

On the other hand, considering the formula [3, p. 344]

$$\Gamma(z+l) = z(z+1)\cdots(z+l-1)\Gamma(z), \tag{63}$$

we have

$$\frac{1}{(\lambda+n/2)\cdots(\lambda+n/2+m-1)} = \frac{\Gamma(\lambda+n/2)}{\Gamma(\lambda+n/2+m)}. \tag{64}$$

From (62), considering (64) and putting $m+1=k$ we have

$$\begin{aligned}\big\langle \big(x_0^2 - |x|^2\big)_-^\lambda, \varphi\big\rangle &= \frac{1}{2^{2k-1}(\lambda+1)\cdots(\lambda+k-1)}\,\frac{\Gamma(\lambda+n/2)}{\Gamma(\lambda+n/2+k-1)}\\ &\quad \times \big\langle \big(x_0^2 - |x|^2\big)_-^{\lambda+k-1}, \Box^{k-1}\varphi\big\rangle.\end{aligned} \tag{65}$$

From (61) and taking into account (65) and (55) we have

$$\begin{aligned}
&\left\langle \frac{\delta^{(k-1)}(x_0-|x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0+|x|)}{|x|^{(n-2)/2}}, \varphi \right\rangle \\
&= \frac{(k-1)!(k-1)!}{(-1)^{k-1}} \frac{1}{2^{2(k-1)}} \lim_{\lambda \to -k} \left\{ \frac{1}{(\lambda+1)\cdots(\lambda+k-1)} \right. \\
&\quad \left. \times \frac{\Gamma(\lambda+n/2)}{\Gamma(\lambda+n/2+k-1)} (\lambda+k)^2 \left\langle \left(x_0^2-|x|^2\right)_-^{\lambda+k-1}, \Box^{k-1}\varphi \right\rangle \right\} \\
&= \frac{\Gamma(n/2-k)(k-1)!}{2^{2(k-1)}\Gamma(n/2-1)} \lim_{\mu \to -1} \left\{ (\mu+1)^2 \left\langle \frac{(x_0^2-|x|^2)_-^{\lambda}}{|x|^{n-2}}, \Box^{k-1}\varphi \right\rangle \right\} \\
&= \frac{\Gamma(n/2-k)(k-1)!}{2^{2(k-1)}\Gamma(n/2-1)} \frac{1}{2} \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \left\langle \delta(x_0, x_1, \ldots, x_{n-1}), \Box^{k-1}\varphi \right\rangle \\
&= \frac{\Gamma(n/2-k)(k-1)!}{2^{2(k-1)}\Gamma(n/2-1)} \frac{1}{2} \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \left\langle \Box^{k-1}\delta(x_0, x_1, \ldots, x_{n-1}), \varphi \right\rangle. \qquad (66)
\end{aligned}$$

From (66) and considering (63) we conclude

$$\begin{aligned}
&\frac{\delta^{(k-1)}(x_0-|x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0+|x|)}{|x|^{(n-2)/2}} \\
&= \frac{\Gamma(n/2-k)(k-1)!(n/2-1)}{2^{2(k-1)}\Gamma(n/2)} \frac{1}{2} \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \Box^{k-1}\left\{\delta(x_0, x_1, \ldots, x_{n-1})\right\}. \qquad (67)
\end{aligned}$$

The formula (67) coincides with the formulae (59) and (60). □

Theorem 1, formulae (59) and (60) generalize the multiplicative product of

$$\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}} \quad \text{and} \quad \frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}}$$

given in [1, formula (4.4), p. 158].

In fact, putting $k=1$ in (59) and considering (60) we have

$$\frac{\delta^{(k-1)}(x_0-|x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0+|x|)}{|x|^{(n-2)/2}} = \frac{1}{2}\pi^{(n-1)/2} \frac{1}{\Gamma((n-1)/2)} \delta(x_0, x_1, \ldots, x_{n-1}). \qquad (68)$$

The formula (68) coincides with the formula (44).

Theorem 2. Let k be a positive integer and n dimension of the space. Then the following formulae are valid:

1. If n is odd,

$$\delta^{(k-1)}\left(x_0-|x|\right) \cdot \delta^{(k-1)}\left(x_0+|x|\right) = 0. \qquad (69)$$

2. If n is even,

$$\delta^{(k-1)}\big(x_0-|x|\big)\cdot\delta^{(k-1)}\big(x_0+|x|\big)=0 \tag{70}$$

if $k\neq n/2, n/2+1,\dots,$ and

$$\delta^{(k-1)}\big(x_0-|x|\big)\cdot\delta^{(k-1)}\big(x_0+|x|\big)=b_{n,k}\Box^{k-n/2}\big\{\delta(x_0,x_1,\dots,x_{n-1})\big\} \tag{71}$$

if $k=n/2+j,\ j=0,1,2,\dots,$ where

$$b_{n,k}=\frac{(-1)^{n/2}\pi^{n/2-1}(k-1)!}{2^{2k-n}(k-n/2)!(-1)^{k-1}}. \tag{72}$$

Proof. From (34) and considering (56) we have

$$\begin{aligned}
&\delta^{(k-1)}\big(x_0-|x|\big)\cdot\delta^{(k-1)}\big(x_0+|x|\big)\\
&\quad=\frac{(k-1)!(k-1)!}{(-1)^{k-1}}\operatorname*{Res}_{\lambda=-k}\big(x_0-|x|\big)_+^{\lambda}\operatorname*{Res}_{\lambda=-k}\big(x_0+|x|\big)_-^{\lambda}\\
&\quad=\frac{(k-1)!(k-1)!}{(-1)^{k-1}}\lim_{\lambda\to-k}(\lambda+k)\big(x_0-|x|\big)_+^{\lambda}\lim_{\lambda\to-k}(\lambda+k)\big(x_0+|x|\big)_-^{\lambda}\\
&\quad=\frac{(k-1)!(k-1)!}{(-1)^{k-1}}\lim_{\lambda\to-k}(\lambda+k)^2\big(x_0^2-|x|^2\big)_-^{\lambda}.
\end{aligned} \tag{73}$$

On the other hand, from (49) and taking into account the conditions

$$\text{(a)}\quad n \text{ odd} \quad \text{and} \tag{74}$$

$$\text{(b)}\quad k\neq n/2, n/2+1,\dots, \tag{75}$$

if n is even we have

$$\lim_{\lambda\to-k}\big\langle(\lambda+k)^2\big(x_0^2-|x|^2\big)_-^{\lambda},\varphi\big\rangle=0. \tag{76}$$

For the case n even and

$$k=n/2, n/2+1,\dots \tag{77}$$

we consider equation (53).

In fact, by writing $k=n/2+j,\ j=0,1,2,\dots,$ and considering (53) we have

$$\begin{aligned}
&\lim_{\lambda\to-k}\big\langle(\lambda+k)^2\big(x_0^2-|x|^2\big)_-^{\lambda},\varphi\big\rangle\\
&\quad=\lim_{\lambda\to-n/2-j}\big\langle(\lambda+n/2+j)^2\big(x_0^2-|x|^2\big)_-^{\lambda},\varphi\big\rangle\\
&\quad=\big\langle c_{-2}^{(j)},\varphi\big\rangle=\frac{(-1)^{n/2}\pi^{n/2-1}}{2^{2j}j!\Gamma(n/2+j)}\Box^{j}\big\{\delta(x)\big\}\\
&\quad=\frac{(-1)^{n/2}\pi^{n/2-1}}{2^{2(k-n/2)}(k-n/2)!(k-1)!}\Box^{k-n/2}\big\{\delta(x)\big\}.
\end{aligned} \tag{78}$$

Therefore, from (73) and considering (74)–(76) and (78) we obtain

$$\delta^{(k-1)}\big(x_0 - |x|\big) \cdot \delta^{(k-1)}\big(x_0 + |x|\big) = 0 \tag{79}$$

if n is odd, and if n is even,

$$\delta^{(k-1)}\big(x_0 - |x|\big) \cdot \delta^{(k-1)}\big(x_0 + |x|\big) = 0 \tag{80}$$

for $k \neq n/2, n/2+1, \ldots$ and

$$\delta^{(k-1)}\big(x_0 - |x|\big) \cdot \delta^{(k-1)}\big(x_0 + |x|\big) = \frac{(-1)^{n/2}\pi^{n/2-1}(k-1)!}{2^{2k-n}(k-n/2)!(-1)^{k-1}}\Box^{k-n/2}\big\{\delta(x)\big\} \tag{81}$$

for $k = n/2, n/2+1, \ldots$.

The formulae (79)–(81) coincide with the formulae (69)–(71), respectively. □

3. Application

As we showed before, theorem 1, formulae (59) and (60) taking $k = 1$ became

$$\frac{\delta^{(k-1)}(x_0 - |x|)}{|x|^{(n-2)/2}} \cdot \frac{\delta^{(k-1)}(x_0 + |x|)}{|x|^{(n-2)/2}} = \frac{1}{2}\pi^{(n-1)/2}\frac{1}{\Gamma((n-1)/2)}\delta(x_0, x_1, \ldots, x_{n-1}),$$

which, together with [2], has an important application of the theory $\lambda\phi^4$, because it allows computation of self-energy, see [1, pp. 159–160].

References

[1] M. Aguirre Téllez, The multiplicative product $[\delta(x_0 - |x|)/|x|^{(n-2)/2}] \cdot [\delta(x_0 + |x|)/|x|^{((n-2)/2)}]$, J. Math. Chem. 22 (1997) 149–160.

[2] C. Bollini and M. Rocca, Convolution of ultradistributions and field theory, Internat. J. Theoret. Phys. 38(9) (1999).

[3] A. Erdelyi, *Transcendental Functions*, Vols. I and II (McGraw-Hill, New York, 1953).

[4] I.M. Gelfand and G.E. Shilov, *Generalized Functions*, Vol. I (Academic Press, New York, 1964).